<u>Calculation of Scroll Length</u>

Here is a description of the derivation of the formula.

Given a scroll with an initial circumference and a specific thickness of the papyrus, calculate the Length of the scroll. A scroll is normally wrapped about a wooden rod. Such a rod would represent the final circumference. In order to maximize the possible length we will assume a rod circumference of 0.

We will use the following notations:

$$C_0 = Initial\ Circumference\ of\ the\ scroll.$$

$$C_f = Final\ Circumference\ or\ Rod\ Circumference$$

$$\Delta C = The\ change\ in\ circumference\ from\ layer\ to\ layer$$

$$D_0 = Initial\ Diameter\ of\ the\ scroll$$

$$D_f = Final\ Diameter, or\ Diameter\ of\ Rod$$

$$T = Thickness\ of\ the\ Papyrus\ fabric$$

$$n = Number\ of\ windings\ on\ the\ Scroll$$

$$L = Length\ of\ papyrus\ removed\ from\ scroll$$

Note that the Diameter and Circumference are related by the equation:

$$C_0 = \pi D_0$$

Note that Thickness and Diameter are related by the equations:

$$D_0 = 2Tn + D_f$$

$$C_0 = 2\pi Tn + \pi D_f$$

$$\Delta C = 2\pi T$$

In other words, the diameter of the Whole is equal to the Diameter of the rod plus twice the sum of the thicknesses of the windings. And the change in circumference is directly related to the thickness of the parchment.

As we peel layers off the scroll, we will get a length of parchment equal to the sum of the circumferences of the windings removed. We could express this with a simple summation:

$$L = \sum_{i=1}^{n} C_i$$

Since circumference is related to diameter we could also write this as

$$L = \sum_{i=1}^{n} \pi D_i \; or \; L = \pi \sum_{i=1}^{n} D_i$$

The first winding will equal the initial diameter of the scroll, and the diameter of the scroll will decrease by 2 thicknesses with every winding removed. So:

$$D_1 = D_0$$

And:

$$D_2 = D_0 - 2T$$

$$D_3 = D_2 - 2T = D_0 - 4T$$

$$D_n = D_{n-1} - 2T = D_0 - 2(n-1)T$$

So:

$$L = \pi \sum_{i=1}^{n} (D_0 - 2(i-1)T)$$

Summations of sums and differences can be split; doing a little work we eventually get the equation:

$$L = \sum_{i=1}^{n} \pi D_0 - (\sum_{i=1}^{n} 2\pi T i - \sum_{i=1}^{n} 2\pi T)$$

The first summation we recognize as the Circumference we started with, the last is the change in circumference; simplifying we get:

$$L = \sum_{i=1}^{n} C_0 - (\Delta C \sum_{i=1}^{n} i - \sum_{i=1}^{n} \Delta C)$$

The summation of a constant is just n times the constant, the first and last values are of this form. The summation of:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

So we get:

$$L = nC_0 - (\Delta C \frac{n(n+1)}{2} - \Delta C n)$$

$$L = nC_0 - \Delta C \left(\frac{n(n+1)}{2} - n\right)$$

$$L = nC_0 - \Delta C \left(\frac{n(n+1) - 2n}{2}\right)$$

$$L = nC_0 - \Delta C\left(\frac{n(n+1-2)}{2}\right)$$

$$L = nC_0 - \Delta C\left(\frac{n(n-1)}{2}\right)$$

So:

$$L = nC_0 - \Delta C\frac{n(n-1)}{2} \; or \; L = nC_0 - 2\pi T\frac{n(n-1)}{2}$$

This equation gives us an easy way to calculate length based upon the initial circumference and the change in circumference. Note that it is easy to put back the 2πT to make the equation in terms of initial circumference and parchment thickness.

But it would be nice to get a calculation for L not based upon n. One that would tell us the max length that could be fit on a roll of specific size just based upon initial circumference and the change in circumference.

I note that the equation as we have it is binomial in nature, i.e. it has an n^2 factor, an n factor, and a constant. This suggests that applying the quadratic formula may give us an approach to solving for n.

I massaged the equation into this form:

$$-L = \pi T n^2 - (C_0 + \pi T)n \;\; OR \;\; 0 = \pi T n^2 - (C_0 + \pi T)n + L$$

Now the equation is in more standard quadratic form of:

$$0 = ax^2 + bx + c$$

In our case:

$$a = \pi T, b = -(C_0 + \pi T), and\ c = L$$

Plugging those values into the quadratic formula we get:

$$n = \frac{(C_0 + \pi T) \pm \sqrt{(C_0 + \pi T)^2 - 4\pi T L}}{2\pi T}$$

The section of the equation under the radical determines whether the formula has real roots. So focusing in on this section I knew that I'd have real roots whenever that section was non-negative. In order for that to be the case I knew that:

$$(C_0 + \pi T)^2 \geq 4\pi T L$$

Solving that equation for L I ended up with:

$$L \leq \frac{(C_0 + \pi T)^2}{4\pi T}$$

Replacing again the:

$$\Delta C = 2\pi T$$

I ended with:

$$L \leq \frac{\left(C_0 + \frac{\Delta C}{2}\right)^2}{2\Delta C}$$

This equation gives us an upper bound for L based solely upon the scroll circumference and the change in circumference from one layer to the next. Of course if you prefer using parchment thickness step back one step to the equations still using T.

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