Calculation of Scroll Length

Here is a description of the derivation of the formula.

Given a scroll with an initial circumference and a specific thickness of the papyrus, calculate the Length of the scroll. A scroll is normally wrapped about a wooden rod. Such a rod would represent the final circumference. In order to maximize the possible length we will assume a rod circumference of 0.

We will use the following notations:

$C_0 = Initial\ Circumference\ of\ the\ scroll.$

$\Delta C = The\ change\ in\ circumference\ from\ layer\ to\ layer$

$T = Thickness\ of\ the\ Papyrus\ fabric$

$L = Length\ of\ papyrus\ removed\ from\ scroll$

Notice that on the last wrap the radius varies between

$$\frac{C_0 \pm \frac{\Delta C}{2}}{2\pi}$$

If the scroll is tightly wrapped the change in the circumference between wraps is related to the papyrus thickness.

$\Delta C = 2\pi T$

The radius of the scroll at any angle follows the well known Archimedes spiral curve which we give here as a function of angle (in radians) where every 2 π increment marks a new revolution.

$$r(\theta) = \frac{\Delta C}{4\pi^2}\theta \qquad (1)$$

Since

$$r_0 = \frac{C_0 + \frac{\Delta C}{2}}{2\pi}$$

then from (1)

$$\theta_0 = \frac{2\pi C_0}{\Delta C} + \pi \qquad (2)$$

The arc length for an equation given in polar coordinates is

$$L = \int \sqrt{r^2(\theta)+\left(\frac{dr}{d\theta}\right)^2} d\theta \quad (3)$$

Plugging in the spiral expression (1) into equation (3) yields

$$L = \frac{\Delta C}{4\pi^2} \int \sqrt{1+\theta^2} d\theta \quad ,$$

which can be found can be resolved by consulting an intergral table:

$$L = \frac{\Delta C}{8\pi^2}\left[\theta\sqrt{1+\theta^2}+\ln\left[\theta+\sqrt{1+\theta^2}\right]\right] \qquad (4)$$

This expression can then be evaluated at the value found in (2). Note that for large values expression (4) L can be approximated by

$$L \leq \frac{\left(C_0+\frac{\Delta C}{2}\right)^2}{2\Delta C}$$

This equation gives us an upper bound for L based solely upon the scroll circumference and the change in circumference from one layer to the next. Of course if you prefer using parchment thickness step back one step to the equations still using T.